CONTINUOUS FUNCTIONS

Def: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then $f: X \to Y$ is said to be **continuous** if $U = f^{-1}(V)$ is open $\forall\, V \subseteq Y$ open.

Eg: 1. Constant functions between any pair of topological spaces are continuous.

Proof: $\forall\, V \subseteq Y$ open, if V contains the const. value $f(X)$, then $f^{-1}(V) = X \in \mathcal{T}_X$. Else $f^{-1}(V) = \phi \in \mathcal{T}_X$.

2. $f: \mathbb{R} \to \mathbb{R}$ with std. topology of the form $f(x) = ax + b$, where $a, b \in \mathbb{R}$ is cts.

Exercise: The pre-image of an open interval under $f(x) = ax+b$ is an open interval.

3. As proved in an earlier lecture,
$f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x+y$ and $g: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$ are continuous.

4. If $(X, \mathcal{T})$ is a discrete top. space, any function $f: X \to Y$ is continuous.

Proof: $\because$ any $A \subseteq X$ is open, $f^{-1}(V)$ is open $\forall\, V \subseteq Y$ open.

Prop. Let P and Q be sets and $f: P \to Q$ be a function.
Then for any topology $\mathcal{T}$ on Q, the collection $f^{-1}(\mathcal{T}) = \{ f^{-1}(V) : V \in \mathcal{T} \}$ is a topology on P.

Prop: Given top. spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, $f: X \to Y$ is continuous $\Leftrightarrow$ $f^{-1}(\mathcal{T}_Y) \subseteq \mathcal{T}_X$

Proof: $\Rightarrow$: If f is cts, $\forall\, V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$.
$\therefore \{ f^{-1}(V) : V \in \mathcal{T}_Y \} \subseteq \mathcal{T}_X$.
$\Rightarrow f^{-1}(\mathcal{T}_Y) \subseteq \mathcal{T}_X$.

$\Leftarrow$: If $f^{-1}(\mathcal{T}_Y) \subseteq \mathcal{T}_X$, then $\forall\, V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X \Rightarrow f$ is continuous.

Prop: Given a topological space $(Y, \mathcal{T}_Y)$ and a function $f: X \to Y$, the topology $\mathcal{T}_X = f^{-1}(\mathcal{T}_Y)$ is the smallest/coarsest topology on X for which f is continuous.

Proof: By the previous proposition, for any topology $\mathcal{T}$ on X for which f is continuous, we have $\mathcal{T}_X \subseteq \mathcal{T}$.

Additionally, clearly f is continuous for $\mathcal{T}_X$.